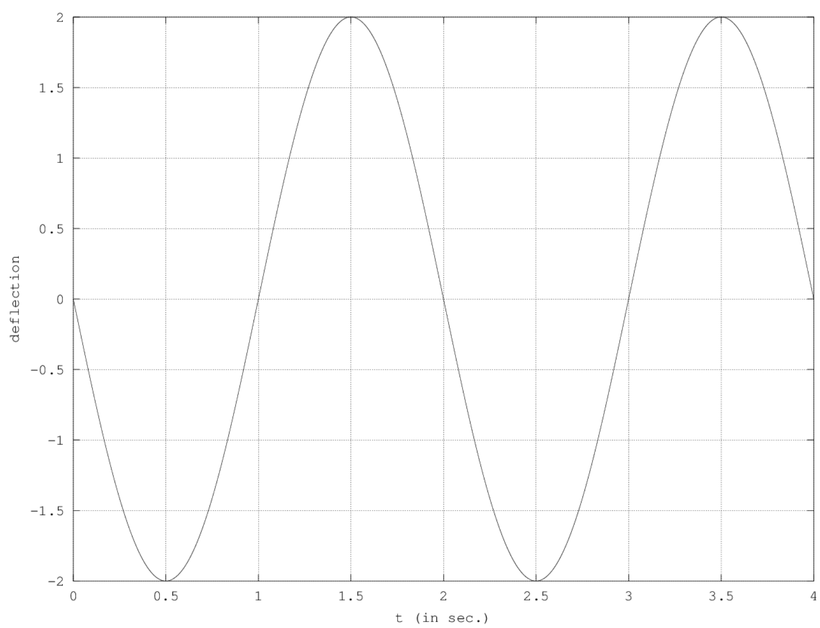
Exam Signals and Systems

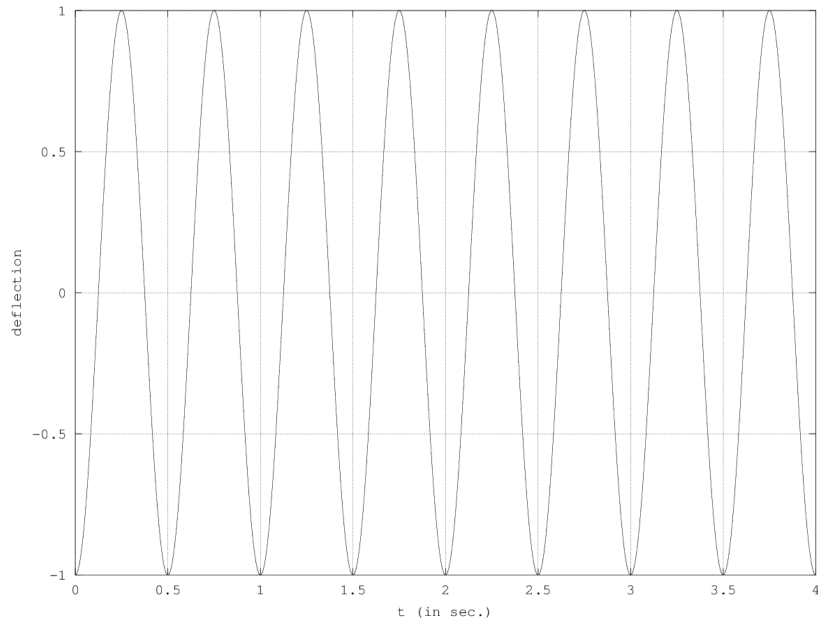
23 januari 2014, 18:30-21:30

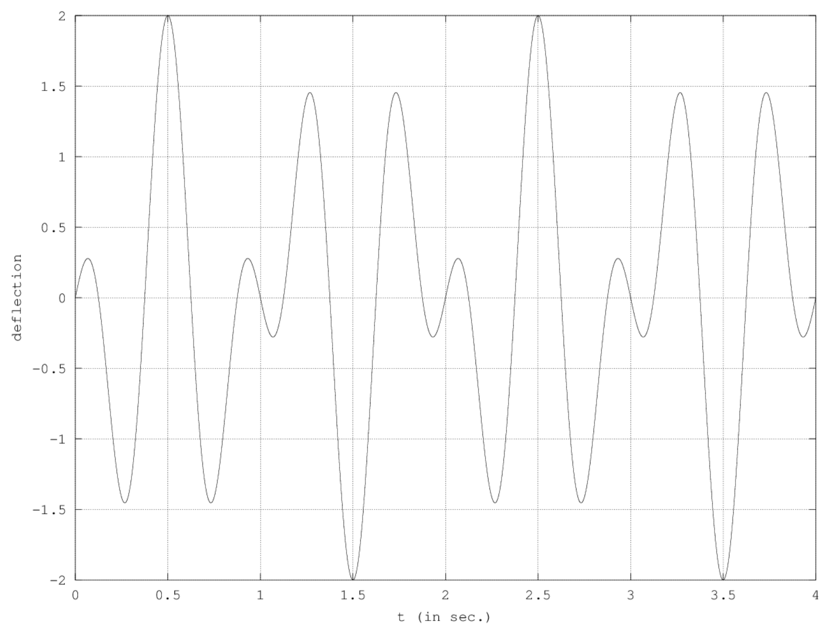
- Write your name, student number and the total number of sheets on the first sheet. Number the sheets!
- You may answer the problems in Dutch or in English.
- Please read each problem fully before making it. Write neatly and carefully. If the handwriting is unreadable, or needs guessing to make something out of it, then the answer is rejected.
- This exam comes with a formula sheet. If you are using a formula from this sheet, please indicate the number of the formula. Other literature, such as the book, may not be consulted.
- The use of a (graphical) calculator is permitted.
- For answers without explanation (even if the answer is correct) no points are awarded.
- The exam consists of four problems. The problems 1, 2 and 3 are worth 20 points each, problem 4 is worth 30 points and you get 10 points for free (total 100 points).

Problem 1: signals and spectra

The figures below show three continuous signals (deflection as a function of the time).







(a) Determine for each signal a formula for the deflection as a function of the time.

(b) Given the continuous signals $x(t)$, $y(t)$ and $z(t)$

$$\begin{aligned} x(t) &= 4\sin(\pi 10t) \\ y(t) &= \cos(\pi 4t + \pi/4) \\ z(t) &= x(t)y(t) \end{aligned}$$

Write each of these signals as a **sum** of terms of the form $Xe^{j\alpha t}$.

(c) Plot the spectrum of each signal. Show the units along the axes and specify for each frequency component the corresponding phase angle.

(d) Make a sketch of the spectrogram of the signal $x(t) = 10\cos(800\pi\sin(2\pi t))$. What do you hear if you listen to this signal?

Problem 2: LTI-systems

(a) Two systems F_0 and F_1 are governed by the difference equations

$$\begin{aligned} y_0[n] &= x[n] \cdot x[n-1] + \cos\left(2\pi(n+1) - \frac{\pi}{3}\right) \\ y_1[n] &= x[n-1] + x[n] + x[n-1] \end{aligned}$$

where $x[n]$ is the input and $y[n]$ is the output. For both systems, determine whether it is *causal*, *linear*, and *time invariant*.

(b) The input-output behaviour of an unknown FIR system F_2 is given in the following table

n	<0	0	1	2	3	≥4
input	0	1	1	1	1	1
output	0	3	4	8	9	14

Determine the impulse response of the system F_2.

(c) Given is the discrete time signal $x[n] = \delta[n] - \delta[n-1] - 2\delta[n-2]$.
The FIR-system F_3 is given by the unit impulse respons

$$h_3[n] = 4\delta[n-1] + 2\delta[n-2] - 4\delta[n-4] - 2\delta[n-5].$$

Determine the output $y[n]$ of the system F_3 if we place $x[n]$ on its input.

(d) Is it possible to construct a FIR system F_4 that produces the output $x[n]$ given the input $F_3\{x[n]\}$? In other words, does there exist a FIR system which is the inverse of F_3? If yes, determine the impulse response of F_4. If not, explain why.

(e) We analyze an unknown FIR system F_5 by placing the above signal $x[n]$ on its input. The system producs the output $y[n] = 3\delta[n-1] - 2\delta[n-2] - 2\delta[n-3] - 5\delta[n-4] - 12\delta[n-5] - 4\delta[n-6]$. Determine the impulse response of the system F_5.

(f) We have five signal processing components with the following impulse responses:

- $c_0[n] = \delta[n-1]$
- $c_1[n] = \delta[n] - \delta[n-1]$
- $c_2[n] = 2\delta[n] + 2\delta[n-1] + 2\delta[n-2]$
- $c_3[n] = 2\delta[n] + \delta[n-1]$
- $c_4[n] = \delta[n] + 2\delta[n-1]$

Is it possible to build the FIR-system F_3 by connecting some (or all) of these components in series? If yes, how? If not, why?

Problem 3: z-transforms

A FIR system is defined by the difference equation

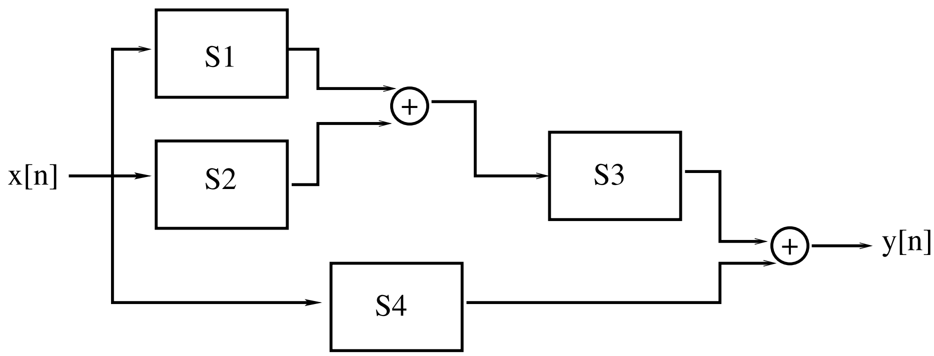
$$y[n] = x[n] - 2\cos(\hat{\omega})x[n-1] + x[n-2].$$

(a) Determine the system function $H(z)$ and the frequency response of this system.

(b) Show that the input $x[n] = A\cos(n\hat{\omega} + \phi)$ will result in an output which is zero everywhere.

(c) Construct the system function $H(z)$ and the difference equation of a system that removes the signal $x[n] = 1 + \cos(\frac{\pi n}{3}) + \cos(\frac{\pi n}{4})$ completely. Of course, the trivial solution $H(z) = 0$ is not allowed.

(d) We consider a connected system that is composed of four FIR systems S_1, S_2, S_3 and S_4.



The system S_1 is defined by the difference equation $y_1[n] = x[n] + x[n-1]$. The system S_2 is defined by the system function $H_2(z) = 1 - z^{-2}$. The system S_3 has the frequency response $H_3(e^{j\hat{\omega}}) = 2e^{-j3\hat{\omega}}$. The system S_4 is defined by the impulse response $h_4[n] = \delta[n-1] + \delta[n-3]$.

Answer the following sub-problems (in any order you like).

- Determine the system function of the connected system.
- Determine the frequency response of the connected system.
- Determine the impulse response of the connected system.
- Determine the difference equation of the connected system.

Problem 4: Fourier analysis

(a) The Fourier coefficients of a continuous signal (with period $T = \frac{1}{2}$ sec.) are:

$$a_k = \begin{cases} 4 & \text{for } k = -3 \\ 2e^{-j\pi/4} & \text{for } k = -1 \\ 5 & \text{for } k = 0 \\ 2e^{j\pi/4} & \text{for } k = 1 \\ 4 & \text{for } k = 3 \\ 0 & \text{for all other } k \end{cases}$$

This signal can be written in the form $DC + A\cos(2\pi f_0 t + \phi_0) + B\cos(2\pi f_1 t + \phi_1)$. Determine the values of DC, A, f_0, ϕ_0, B, f_1 and ϕ_1.

(b) Determine the Fourier coefficients of the function $x(t) = 1 + 2\cos(2\pi 6t) + 3\sin(2\pi 9t)$.

(c) Given is a periodic continous signal $s(t)$ with period T_0. The interval $[0, T_0]$ is split in five sub-intervals $[0, aT_0]$, $[aT_0, bT_0]$, $\ldots$, $[dT_0, T_0]$, where $0 \leq a \leq b \leq c \leq d \leq 1$.

$$s(t) = \begin{cases} 0 & \text{for } 0 \leq t < aT_0 \\ 1 & \text{for } aT_0 \leq t < bT_0 \\ 0 & \text{for } bT_0 \leq t < cT_0 \\ -1 & \text{for } cT_0 \leq t < dT_0 \\ 0 & \text{for } dT_0 \leq t < T_0 \end{cases}$$

Show that the Fourier coefficients a_k of the signal $s(t)$ are given by

$$a_k = \begin{cases} b + c - a - d & \text{for } k = 0 \\ \frac{-1}{j2\pi k}\left(e^{-j2\pi kb} - e^{-j2\pi ka} + e^{-j2\pi kc} - e^{-j2\pi kd}\right) & \text{for } k \neq 0 \end{cases}$$

(d) The signal $x(t)$ (with period 6 sec.) is defined by:

$$x(t) = \begin{cases} 1 & \text{for } 0 \leq t < 3 \\ -1 & \text{for } 3 \leq t < 6 \end{cases}$$

Show that the Fourier coefficients a_k of the signal $x(t)$ are

$$a_k = \begin{cases} 0 & \text{for } k = 0 \\ 0 & \text{for } k \text{ even} \\ \frac{2}{j\pi k} & \text{for } k \text{ odd} \end{cases}$$

(e) The periodic signal $y(t)$ (with period 6 sec.) is defined by:

$$y(t) = \begin{cases} 1 & \text{for } 0 \leq t < 1 \\ 2 & \text{for } 1 \leq t < 2 \\ 1 & \text{for } 2 \leq t < 3 \\ -1 & \text{for } 3 \leq t < 4 \\ -2 & \text{for } 4 \leq t < 5 \\ -1 & \text{for } 5 \leq t < 6 \end{cases}$$

Determine the Fourier coefficients a_k of the signal $y(t)$.

Formula sheet Signals and Systems

$$\cos(\theta) = \sin(\theta + \pi/2) \quad (1)$$
$$\cos(\theta) = \cos(\theta + 2\pi k) \text{ for integer } k \quad (2)$$
$$\cos(\theta) = \cos(-\theta) \quad (3)$$
$$\sin(-\theta) = -\sin(\theta) \quad (4)$$
$$\cos(2\pi k) = 1 \text{ for integer } k \quad (5)$$
$$\cos(\pi k + \frac{\pi}{2}) = 0 \text{ for integer } k \quad (6)$$
$$\cos(2\pi k + \pi) = -1 \text{ for integer } k \quad (7)$$
$$j^2 = -1 \quad (8)$$
$$Re(a + jb) = a \quad (9)$$
$$Im(a + jb) = b \quad (10)$$
$$(a + jb)^* = a - jb \quad (11)$$
$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \quad (12)$$
$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad (13)$$
$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad (14)$$
$$\int e^{\alpha t} dt = \frac{1}{\alpha} e^{\alpha t} + c \quad (15)$$
$$\int t e^{\alpha t} dt = \frac{\alpha t - 1}{\alpha^2} e^{\alpha t} + c \quad (16)$$
$$x[n] = x(nT_s) \text{ Perfect A-to-D conversion} \quad (17)$$
$$f_s = \frac{1}{T_s} \text{ Sampling frequency} \quad (18)$$
$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s} \text{ Normalized Radian Frequency} \quad (19)$$
$$y(t) = \sum_{n=-\infty}^{\infty} y[n] p(t - nT_s) \text{ Interpolation/Reconstruction} \quad (20)$$
$$p(t) = \frac{\sin(\pi t/T_s)}{\pi t/T_s}, \quad -\infty < t < \infty \text{ Sinc pulse} \quad (21)$$
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt} \text{ Fourier synthesis} \quad (22)$$
$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} \, dt \text{ Fourier analysis} \quad (23)$$
$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn} \text{ DFT (Discrete Fourier Transform)} \quad (24)$$
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn} \text{ Inverse DFT} \quad (25)$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \text{ FIR system} \tag{26}$$

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] \text{ Unit impulse response} \tag{27}$$

$$y[n] = \sum_{k=0}^{M} h[k] x[n-k] \text{ Convolution sum FIR-system} \tag{28}$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \text{ General convolution} \tag{29}$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} = \sum_{k=0}^{M} h[k] e^{-j\hat{\omega}k} \text{ Frequency response FIR-system} \tag{30}$$

$$h_1[n] * h_2[n] \leftrightarrow H_1(e^{j\hat{\omega}}) H_2(e^{j\hat{\omega}}) \tag{31}$$

$$D_L(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}L/2)}{L \sin(\hat{\omega}/2)} \text{ Dirichlet function} \tag{32}$$

$$X(z) = \sum_{k=0}^{N} x[k] z^{-k} \text{ Z-transform} \tag{33}$$

$$H(z) = \sum_{k=0}^{N} b_k z^{-k} = \sum_{k=0}^{N} h[k] z^{-k} \text{ System function FIR system} \tag{34}$$

$$a x_1[n] + b x_2[n] \leftrightarrow a X_1(z) + b X_2(z) \text{ Linearity of z-transform} \tag{35}$$

$$y[n] = h[n] * x[n] \leftrightarrow Y(z) = H(z) X(z) \text{ Convolution via z-domain} \tag{36}$$

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \tag{37}$$